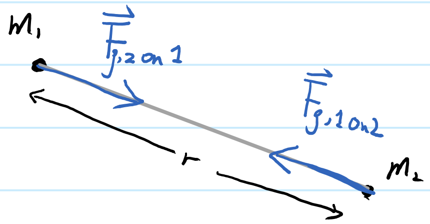
Lecture 12 Gravitation I

Newton's Law of gravity (inverse square law)

- Massive particles attract each other.



- magnitude of the force

$$F_g \propto \begin{cases} m_1 \\ m_2 \\ 1/r^2 \end{cases} \Rightarrow \boxed{F_g = G\frac{m_1 m_2}{r^2}}$$

why not $\frac{1}{r^3}, \frac{1}{r^4}, \frac{1}{r^{2.01}}$?

universal for everything in the universe!

universal constant.

G: gravitational constant (big G)

- direction: always pointing towards the other mass.

- Weakest among four fundamental forces.

$$G = 6.671 \times 10^{-11} \ N \, m^2/kg^2$$

example: Force between 2 apples (~100 kg) separated by 1 m

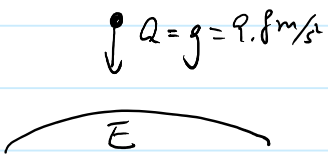
$$F_g = \frac{G \cdot m \cdot m}{r^2} = 6.671 \times 10^{-13} \ N.$$

very small

Mass of Earth. — how to measure it?

Henry Cavendish measured G using a torsion balance.

We know acc. due to gravity on Earth is $9.8 \, m s^{-2}$, and radius of

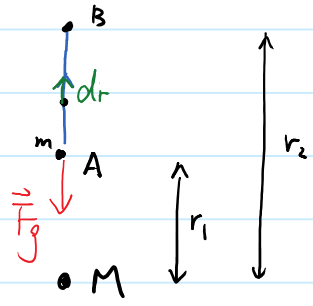


$$\underset{\frac{G M_E m}{R_E^2}}{F} = ma = mg \Rightarrow M_E = \frac{R_E^2 g}{G} = 5.97 \times 10^{24} kg$$

Gravitational Potential Energy (beyond mgh)

$$\Delta U_{A\to B} = -\int_A^B \vec{F}_g \cdot d\vec{r} = -\int_{r_1}^{r_2} -\frac{GMm}{r^2} \cdot dr$$

$d\vec{r}$ & $\vec{F}_g$ are opposite.



$$= \int_{r_1}^{r_2} \frac{GMm}{r^2} dr$$

$$= -\frac{GMm}{r}\Big|_{r_1}^{r_2}$$

$$\Delta U_{r_1\to r_2} = -\frac{GMm}{r_2} + \frac{GMm}{r_1}$$

Potential Energy at r :

$$U(r) = U_{ref} + \Delta U_{ref\to r}$$

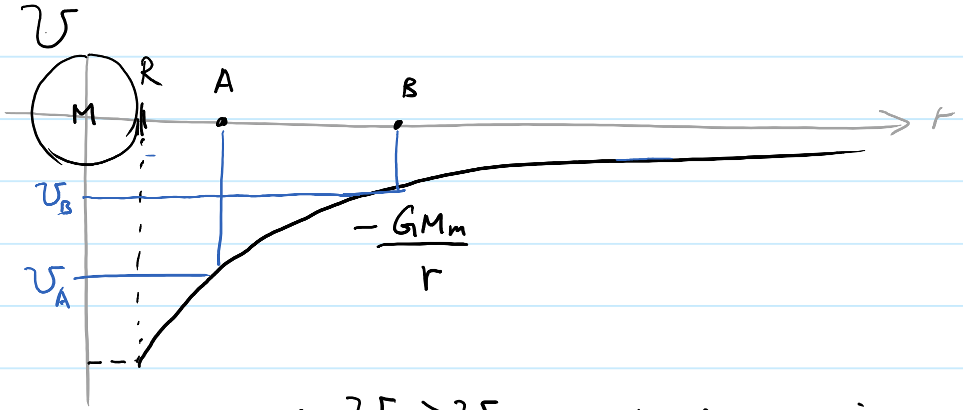
$r_2 \to r$
$r_1 \to r_{ref}$

$$= U_{ref} - \frac{GMm}{r} + \frac{GMm}{r_{ref}}$$

Both the position of the reference point and the value of U_{ref}, potential energy at the reference point, are free to choose.

We will choose $U_{ref} = 0$ and $r_{ref} \to \infty$ to be our reference so that $U(r)$ is simplified as

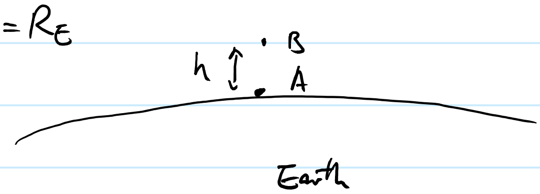
$$\boxed{U(r) = -\frac{GMm}{r}} \quad \text{given } U(\infty) = 0$$



• $U_B > U_A$ moving up increases U. ✓

Recovering "mgh"

Suppose $r_B - r_A = h$ and $h \ll r_A = R_E$



$$\Delta U_{A\to B} = -\frac{GMm}{R_E + h} + \frac{GMm}{R_E}$$

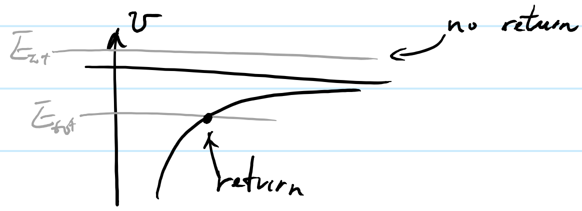
$$= \frac{GMm \quad h}{R_E\,(R_E + h)} \qquad R_E + h \simeq R_E$$

$$\simeq \frac{GMm}{R_E^2}\,h$$

$$= mg \cdot h \longrightarrow$$ mgh is the approximation of ΔU for $h \ll R_E$.

Escape Velocity

- Min. velocity required to move from the surface of a planet to infinitely far away.



$E_{tot} \geq 0$

$$K + U \geq 0 \Rightarrow \frac{1}{2}mv^2 - \frac{GMm}{R} \Rightarrow V \geq \sqrt{\frac{2GM}{R}}$$

For earth, $V_{esc} = 1.12 \times 10^4$ m/s.

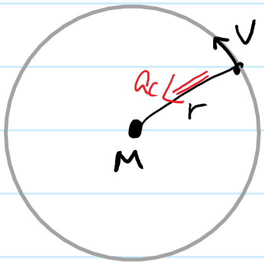
Satellite (circular orbit)

Properties of circular orbit:

① v is constant $\Rightarrow$ K is constant
② r is constant $\Rightarrow$ U is constant.

$\Big\}$ E_{tot} is constant.

③ Ratio of K/U is constant.



$$\frac{GMm}{r^2} = \underline{F_g = ma} = \frac{mv^2}{r}$$

$$\Rightarrow \frac{GM}{r} = v^2 \Rightarrow v = \sqrt{\frac{GM}{r}}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{GM}{r} = -\frac{1}{2}U$$

$$U = -\frac{GMm}{r}$$

$$\Rightarrow \frac{K}{U} = -\frac{1}{2}, \quad E_{tot} = K + U = -\frac{1}{2}U + U = \frac{1}{2}U$$

$$\boxed{E_{tot} = -\frac{GMm}{2r}} < 0 \Longrightarrow \text{Bounded orbit}$$

the radius of the orbit alone determines the total energy.

period, T

$$vT = 2\pi r$$

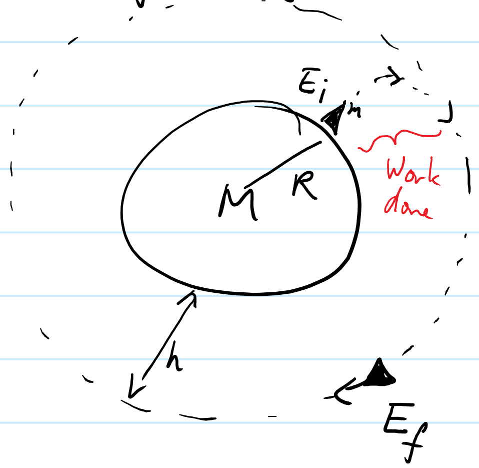
$$T = \frac{2\pi r}{v} = \frac{2\pi}{\sqrt{GM}} r^{3/2} \quad \propto r^{3/2}$$

Kepler's 2nd Law.

Example.

How much work is needed to fire a rocket to a circular orbit at height, h, above the ground?

Initially, the rocket is rest on the surface of the Earth.



$$E_i = \cancel{K_i}^{0} + U_i = -\frac{GMm}{R}$$

on the orbit.

$$E_f = -\frac{GMm}{2(R+h)}$$

Work-energy theorem.

$$W_{engine} = \Delta E_{tot} = E_f - E_i$$

$$W_{engine} = GMm\left[\frac{1}{R} - \frac{1}{2(R+h)}\right]$$